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MULTIFREQUENCY RESONANCE OSCILLATIONS UNDER EXTERNAL PERTURBANCES

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Multifrequency oscillations in systems with a large number of degrees of freedom were considered in [1, 2]. In the present paper we study multifrequency oscillations of systems of a more specific form; we reduce the problem to the study of canonical systems of differential equations describing the resonance phenomena.

1. We consider a conservative system with n degrees of freedom, which has a stable position of equilibrium; in a neighborhood of this position the system performs relatively small oscillations. The system is acted on by N perturbations, which neither change the position of equilibrium nor lead out the motion of the system beyond the neighborhood of this position. We shall regard these perturbations as generalized coordinates (with index larger than n), which are specified functions of time. These coordinates enter formally into the expressions for the kinetic and potential energies (i.e. we assume that the conditional system with $n+N$ coordinates is a conservative system). We assume also that owing to a specified internal symmetry in the system, the expressions for the kinetic and potential energies are symmetric with respect to all of the $n+N$ generalized coordinates. Then

$$T=\frac{1}{2}\sum_{i,k=1}^{n+N} A_{ik}q_i^{\cdot}q_k^{\cdot}, \quad A_{ik}=a_{ik}+\frac{1}{2}\sum_{j,s=1}^{n+N} a_{ik}^{(js)}q_jq_s+\ldots \tag{1.1}$$

$$\Pi=\frac{1}{2}\sum_{i,k=1}^{n+N} c_{ik}q_iq_k+\frac{1}{24}\sum_{i,k,j,s=1}^{n+N} c_{ik}^{(js)}q_iq_kq_jq_s+\ldots \quad (c_{ik}=c_{ki}, c_{ik}^{(js)}=c_{jk}^{(is)}=\ldots,\ldots)$$

We assume that the symmetry of the coefficients in the expressions for Π, shown in the parentheses, holds also for the coefficients in the expression for T, i.e.

$$a_{ik}=a_{ki}, \qquad a_{ik}^{(js)}=a_{jk}^{(is)}=\ldots,\ldots \tag{1.2}$$

This assumption, without restricting the generality of results, leads to more simple and symmetric relationships.

We obtain the differential equations of motion from the equations (1.1) upon using the relations (1.2) and the fact that the q_i are known functions of time for $i=n+1,\ldots,n+N$.

Let us assume that all perturbations are harmonic with frequencies $p_j (j=1,2,\ldots,N)$.

Then in the equations of motion we set

$$q_k = \varepsilon^{1/2} q_k', \quad k = 1, 2, \ldots, n, \quad q_{n+j} = \varepsilon^{1/2} H_j \cos(p_j t - \psi_j), \quad j = 1, 2, \ldots, N$$

where ε is a small positive parameter. Omitting the primes, we obtain, to within terms of the third order,

$$\sum_{k=1}^{n} (a_{ik} q_k^{\cdot\cdot} + c_{ik} q_k) = \sum_{k=1}^{N} (a_{i,\, n+k} p_k^2 - c_{i,\, n+k}) H_k \cos(p_k t - \psi_k) - \varepsilon F_i^*, \qquad i = 1, 2, \ldots, n \tag{1.3}$$

$$F_i^* = \sum_{k,\, j,\, s=1}^{n} \left[\frac{1}{2} a_{ik}^{(js)} (q_k q_j q_s^{\cdot\cdot} + q_k q_j^{\cdot} q_s^{\cdot}) + \frac{1}{6} c_{ik}^{(js)} q_k q_j q_s\right] +$$

$$\frac{1}{2} \sum_{k,\, j=1}^{n} \sum_{s=1}^{N} H_s \{\cos(p_s t - \psi_s) [a_{ik}^{(j,\, n+s)} (2 q_k q_j^{\cdot\cdot} - p_s^2 q_k q_j + q_k^{\cdot} q_j) +$$

$$c_{ik}^{(j,\, n+s)} q_k q_j] - \sin(p_s t - \psi_s) a_{ik}^{(j,\, n+s)} p_s q_k q_j^{\cdot}\} + \frac{1}{4} \sum_{k=1}^{n} \sum_{j,\, s=1}^{N} H_j H_s \times$$

$$\{\cos[p_j + p_s) t - \psi_j - \psi_s] [a_{ik}^{(n+j,\, n+s)} (q_k^{\cdot\cdot} - 2 p_s^2 q_k - p_j p_s q_k) + c_{ik}^{(n+j,\, n+s)} q_k] +$$

$$\cos[(p_j - p_s) t - \psi_j + \psi_s] [a_{ik}^{(n+j,\, n+s)} (q_k^{\cdot\cdot} - 2 p_s^2 q_k + p_j p_s q_k) +$$

$$c_{ik}^{(n+j,\, n+s)} q_k] - 2 a_{ik}^{(n+j,\, n+s)} p_j q_k^{\cdot} [\sin((p_j + p_s) t - \psi_j - \psi_s) +$$

$$\sin((p_j - p_s) t - \psi_j + \psi_s)]\} + \frac{1}{24} \sum_{k,\, j,\, s=1}^{N} H_k H_j H_s \{[c_{i,\, n+k}^{(n+j,\, n+s)} -$$

$$(p_k^2 + p_j^2 + p_s^2 + p_k p_j - p_j p_s - p_s p_k) a_{i,\, n+k}^{(n+j,\, n+s)}] \cos[(p_k + p_j - p_s) t -$$

$$\psi_k - \psi_j + \psi_s] + [c_{i,\, n+k}^{(n+j,\, n+s)} - (p_k^2 + p_j^2 + p_s^2 + p_k p_j + p_j p_s + p_s p_k) \times$$

$$a_{i,\, n+k}^{(n+j,\, n+s)}] \cos[(p_k + p_j + p_s) t - \psi_k - \psi_j - \psi_s]\}$$

Resonance in the system (1.3) holds when one of the following inequalities is satisfied (either exactly or approximately):

$$\sum_{i=1}^{n} q_i^{(k)} \omega_i + \sum_{j=1}^{N} s_j^{(k)} p_j = 0, \quad k = 1, 2, \ldots$$

where $\omega_i (i = 1, 2, \ldots, n)$ are the natural frequencies of the linear part of the system (1.3), and $q_i^{(k)}$, $s_j^{(k)}$ are integers (some of which may, in fact, be zero).

The resonance we consider here arises from the terms of the third degree in the system (1.3), i.e. resonance of the third rank.

We assume now that in the system in question we have m-frequency oscillations with the frequencies $\omega_1 \leqq \omega_2 \ldots \leqq \omega_m$, $m \leqq n$. Further, we shall assume that

$$a_{i,\, n+k} p_k^2 - c_{i,\, n+k} = 0, \qquad k = 1, 2, \ldots N \tag{1.4}$$

We show at the close of our paper that if these relationships are not satisfied, then the calculations become much more cumbersome, however, qualitatively no new results in the study of resonance phenomena are obtained. This was shown, in particular, in [3] for the case of single-frequency oscillations with a single external perturbation.

2. We seek a solution of the system (1.3) in the form

$$q_k = \sum_{s=1}^{m} L_k^{(s)} A_s \cos(\omega_s t - \varphi_s) + \varepsilon q_{k1} + \varepsilon^2 q_{k2} + \dots, \quad k = 1, 2, \dots n \tag{2.1}$$

$$\sum_{k=1}^{n} (c_{ik} - \omega_s^2 a_{ik}) L_k^{(s)} = 0 \quad i, s = 1, 2, \dots, n \tag{2.2}$$

$$\sum_{i,k=1}^{n} a_{ik} L_i^{(j)} L_k^{(s)} = 0, \quad \sum_{i,k=1}^{n} {}'c_{ik} L_i^{(j)} L_k^{(s)} = 0, \quad j \neq s \tag{2, 3}$$

Here A_s and φ_s are slowly varying functions of time, and the $q_{k1}, \dots$ are relatively rapidly varying functions of A_s and φ_s and the time, which supplement the fundamental solution. The functions $L_k^{(s)}$ $(k, s = 1, 2, \dots n)$ are determined from the algebraic systems (2.2) and possess the orthogonality property (2.3).

For the study of multifrequency solutions of system (1.3) let us generalize a supplement (see [4, 5]) to the existing asymptotic methods in the theory of nonlinear oscillations [6]. We substitute equations (2.1) into (1.3), multiply the i th equation by $L_i^{(r)}$, $r = 1, 2, \dots m$, and then add all the equations. Taking (2.3) into account, we obtain

$$(A_r^{\cdot\cdot} + 2\omega_r A_r \varphi_r^{\cdot} - A_r \varphi_r^{\cdot 2}) \cos(\omega_r t - \varphi_r) + (A_r \varphi_r^{\cdot\cdot} - 2\omega_r A_r^{\cdot} + 2A_r^{\cdot} \varphi_r^{\cdot}) \sin(\omega_r t - \varphi_r) + \frac{\varepsilon}{m_r} \sum_{i,k=1}^{n} L_i^{(r)} (a_{ik} q_{k1}^{\cdot\cdot} + c_{ik} q_{k1}) = -\frac{\varepsilon}{m_r} F_r \tag{2.4}$$

Here

$$m_r = \sum_{i,k=1}^{n} a_{ik} L_i^{(r)} L_k^{(r)} > 0$$

$$\begin{aligned}
F_r = {} & \frac{1}{24} \sum_{j,s,u=1}^{m} A_j A_s A_u \{3\bar{g}_{rj}^{(su)}(\omega_j, \omega_s, -\omega_u) \cos[(\omega_j + \omega_s - \omega_u)t - \varphi_j - \\
& \varphi_s + \varphi_u] + \bar{g}_{rj}^{(su)}(\omega_j, \omega_s, \omega_u) \cos[(\omega_j + \omega_s + \omega_u)t - \varphi_j - \varphi_s - \varphi_u]\} + \\
& \frac{1}{8} \sum_{j=1}^{N} \sum_{s,u=1}^{m} H_j A_s A_u \{2\bar{g}_{r,n+j}^{(su)}(\omega_s, -\omega_u, p_j) \cos[(\omega_s - \omega_u + p_j)t - \\
& \varphi_s + \varphi_u - \psi_j] + \bar{g}_{r,n+j}^{(su)}(\omega_s, \omega_u - p_j) \cos[(\omega_s + \omega_u - p_j)t - \varphi_s - \varphi_u + \\
& \psi_j] + \bar{g}_{r,n+j}^{(su)}(\omega_s, \omega_u, p_j) \cos[(\omega_s + \omega_u + p_j)t - \varphi_s - \varphi_u - \psi_j]\} + \\
& \frac{1}{84} \sum_{j,s=1}^{N} \sum_{u=1}^{m} H_j H_s A_u \{2\bar{g}_{r,n+j}^{(n+s,u)}(p_j, \omega_u - p_s) \cos[(\omega_u + p_j - p_s)t - \varphi_u - \\
& \psi_j + \psi_s] + \bar{g}_{r,n+j}^{(n+s,u)}(p_j, p_s, -\omega_u) \cos[(p_j + p_s - \omega_u)t - \psi_j - \psi_s + \varphi_u + \\
& \bar{g}_{r,n+j}^{(n+s,u)}(p_j, p_s, \omega_u) \cos[(p_j + p_s + \omega_u)t - \psi_j - \psi_s - \varphi_u]\} + \\
& \frac{1}{24} \sum_{j,s,u=1}^{N} H_j H_s H_u \{3\bar{g}_{r,n+j}^{(n+s,n+u)}(p_j, p_s, -p_u) \cos[(p_j + p_s - p_u)t - \psi_j - \\
& \psi_s + \psi_u] + \bar{g}_{r,n+j}^{(n+s,n+u)}(p_j, p_s, p_u) \cos[(p_j + p_s + p_u)t - \psi_j - \psi_s - \psi_u]\}
\end{aligned}$$

$$\bar{g}_{ru}^{(sv)}(\omega_u, \omega_s, \omega_v) = g_{ru}^{(sv)} - (\omega_u^2 + \omega_s^2 + \omega_v^2 + \omega_u\omega_s + \omega_s\omega_v + \omega_v\omega_u) h_{ru}^{(sv)}$$

$$h_{ru}^{(pv)} = \sum_{i,k,j,s=1}^{n} a_{ik}^{(js)} L_i^{(r)} L_k^{(u)} L_j^{(p)} L_s^{(v)}, \qquad g_{ru}^{(pv)} = \sum_{i,k,s,j=1}^{n} c_{ik}^{(js)} L_i^{(r)} L_k^{(u)} L_j^{(p)} L_s^{(v)}$$

$$h_{ru}^{(p,\,n+s)} = \sum_{i,k,j=1}^{n} a_{ik}^{(j,\,n+s)} L_i^{(r)} L_k^{(u)} L_j^{(p)}, \quad g_{ru}^{(p,\,n+s)} = \sum_{i,k,j=1}^{n} c_{ik}^{(j,\,n+s)} L_i^{(r)} L_k^{(u)} L_j^{(p)}$$

$$h_{ru}^{(n+j,\,n+s)} = \sum_{i,k=1}^{n} a_{ik}^{(n+j,\,n+s)} L_i^{(r)} L_k^{(u)}, \quad g_{ru}^{(n+j,\,n+s)} = \sum_{i,k=1}^{n} c_{ik}^{(n+j,\,n+s)} L_i^{(r)} L_k^{(u)}$$

$$h_{r,\,n+u}^{(n+j,\,n+s)} = \sum_{i=1}^{n} a_{i,\,n+u}^{(n+j,\,n+s)} L_i^{(r)}, \quad g_{r,\,n+u}^{(n+j,\,n+s)} = \sum_{i=1}^{n} c_{i,\,n+u}^{(n+j,\,n+s)} L_i^{(r)}$$

It follows from the resulting system that resonance of the third rank is possible if for each ω_r we can find $\omega_s, \omega_j, \omega_u, \ldots, p_s, p_j, p_u, \ldots$, such that one of the following equalities is satisfied (exactly or approximately):

$$\begin{aligned}
&\pm\omega_r = \omega_j + \omega_s - \omega_u, \quad \omega_r = \omega_{j_1} + \omega_{s_1} + \omega_{u_1} \\
&\pm\omega_r = \omega_{s_2} - \omega_{u_2} + p_j, \quad \pm\omega_r = \omega_{s_3} + \omega_{u_3} - p_{j_1}, \quad \omega_r = \omega_{s_4} + \omega_{u_4} + p_{j_2} \\
&\pm\omega_r = \omega_{u_5} + p_{j_3} - p_s, \quad \pm\omega_r = p_{j_4} + p_{s_1} - \omega_{u_6}, \quad \omega_r = \omega_{u_7} + p_{j_5} + p_{s_2} \\
&\pm\omega_r = p_{j_6} + p_{s_3} - p_u, \quad \omega_r = p_{j_7} + p_{s_4} + p_{u_1}
\end{aligned} \tag{2.5}$$

The number of possible resonances is bounded, although very large.

Among the first equations of (2.4) there are the trivial ones, i.e.

$$\omega_r = \omega_r + \omega_s - \omega_s, \qquad s = 1, 2, \ldots m \tag{2.6}$$

Except these equations, the first equations of (2.5) define the existence of an internal resonance. In [7] a case of this kind was considered for m-frequency oscillations of the third rank for a conservative system with n degrees of freedom. We shall assume here that the system (1.3) has no internal resonances, i.e. from the first two equations of (2.5) only equations of the form (2.6) are satisfied, and the resonance appears only due to the external perturbations. For situations in which this condition is violated, it is necessary to combine our results with those given in [7].

3. We assume now that the frequencies $\omega_i (i = 1, 2, \ldots, n)$ are all distinct. We then have the possibility of the following six fundamental cases of resonance.

The first fundamental case is characterized by the fact that in the equations defining resonance only one frequency is involved and the simplest canonical systems are obtained. Several subcases are identifiable here.

A. Suppose that $p_r \approx 3\omega_r (r = 1, 2, \ldots, m)$. Here, as in the similar cases given below, we assume that there are no other relations leading to satisfaction of Eqs.(2.5). Using the identity

$$\cos[(p_r - 2\omega_r)t - \psi_r + 2\varphi_r] = \cos\lambda_r \cos(\omega_r t - \varphi_r) - \sin\lambda_r \times \sin(\omega_r t - \varphi_r) \quad (\lambda_r = (p_r - 3\varphi_r)\,t - \psi_r + 3\varphi_r)$$

and equating corresponding terms in the system (2.4) which appear in front of the expressions $\cos(\omega_r t - \varphi_r)$ and $\sin(\omega_r t - \varphi_r)$, we find

$$A_r^{\cdot\cdot} + 2\omega_r A_r \varphi_r^{\cdot} - A_r \varphi_r^{\cdot 2} = -\frac{\varepsilon}{m_r}\left\{\frac{1}{8}\,\bar{g}_{rr}^{(rr)}(\omega_r, \omega_r - \omega_r)\, A_r^3 + \right. \tag{3.1}$$

$$\frac{1}{4}\sum_{s=1}^{m}{}' \bar{g}_{rr}^{(ss)}(\omega_r, \omega_s, -\omega_s) A_r A_s^2 + \frac{1}{42}\sum_{s=1}^{N} \bar{g}_{n+s,n+s}^{(rr)}(\omega_r, p_s, -p_s) A_r H_s^2 +$$
$$\left.\frac{1}{84}\bar{g}_{r,n+r}^{(rr)}(\omega_r, \omega_r, -p_r) H_r A_r^2 \cos\lambda_r\right\}, \qquad A_r\varphi''_r - 2\omega_r A'_r + 2A'_r\varphi'_r =$$
$$\frac{\varepsilon}{m_r}\frac{1}{84}\bar{g}_{r,n+r}^{(rr)}(\omega_r, \omega_r, -p_r) H_r A_r^2 \sin\lambda_r, \qquad r = 1, 2, \ldots, m$$

where the prime on the summation sign means that the term corresponding to $s = r$ is omitted.

From Eqs. (3.1) we obtain, upon taking into account the expression for λ_r to within terms of order ε, the following autonomous system

$$dA_r/\varepsilon dt = -H_r^\circ A_r^2 \sin\lambda_r, \qquad r = 1, 2, \ldots, m \tag{3.2}$$
$$\frac{d\lambda_r}{\varepsilon dt} = 2m_r^\circ H_r^\circ + 3H_r^\circ\alpha_{rr}A_r^2 + 6H_r^\circ\sum_{s=1}^{m}{}'\alpha_{sr}A_s^2 - 3H_r^\circ A_r\cos\lambda_r$$
$$2m_r^\circ H_r^\circ = \frac{p_r - 3\omega_r}{\varepsilon} + 3H_r^\circ\sum_{s=1}^{N}\beta_{sr}H_s^2, \qquad \overline{H}_r = \frac{1}{16m_r\omega_r}\bar{g}_{r,n+r}^{(rr)}(\omega_r\omega_r, -p_r) H_r$$
$$H_r^\circ\alpha_{sr} = -\frac{1}{16m_r\omega_r}\bar{g}_{rr}^{(ss)}(\omega_r, \omega_s, -\omega_s), \qquad H_r^\circ\beta_{sr} = -\frac{1}{8m_r\omega_r}g_{r,n+s}^{(r,n+s)}(\omega_r, p_s, -p_s)$$

The system (3.2) provides a complete representation of all the motions for the resonance case in question. We refer to such systems as canonical systems.

After equating the remaining terms in (2.4), we obtain m equations for q_{k+1} ($k = 1, 2, \ldots n$). The remaining $n - m$ equations are obtained from the system (1.3) following similar procedure. In accord with the method of Krylov-Bogoliubov, these functions are to be obtained from a second approximation, the equations which we shall not consider here.

B. Suppose that $\omega_r \approx 3p_r$ ($r = 1, 2, \ldots, m$). Using the method indicated above, we obtain the canonical system

$$dA_r/\varepsilon dt = H_r^0 \sin\lambda_r, \qquad r = 1, 2, \ldots, m \tag{3.3}$$
$$d\lambda_r/\varepsilon dt = 2m_r^\circ H_r^\circ - \alpha_{rr}H_r^\circ A_r^2 - 2H_r^\circ\sum_{s=1}^{N}{}'\alpha_{sr}A_s^2 + \frac{H_r^\circ}{A_r}\cos\lambda_r$$
$$2m_r^\circ H_r^\circ = \frac{\omega_r - 3p_r}{\varepsilon} - H_r^\circ\sum_{s=1}^{N}\beta_{sr}H_r^2, \qquad H_r^\vee = \frac{1}{16m_r\omega_r}\bar{g}_{r,n+r}^{(n+r,n+r)}(p_r, p_r, p_r) H_r^3$$
$$\lambda_r = (\omega_r - 3p_r)t - \varphi_r + 3\psi_r$$

To this subcase we can add the following:

$$\omega_r \approx 2p_{2r-1} \pm p_{2r}, \quad \omega_r \approx p_{3r-2} \pm p_{3r-1} \pm p_{3r}, \quad r = 1, 2, \ldots, m$$

Here again we obtain the canonical system (3.3) with certain changes in the notation.

4. The second fundamental case is defined by relationships involving two frequencies ω_i. Here also we may identify several subcases.

A. The equations

$$\omega_{2r-1} \approx 2\omega_{2r} - p_r, \quad \omega_{2r} \approx -\omega_{2r} + \omega_{2r-1} + p_r, \quad r = 1, 2, \ldots, m$$

are satisfied. It is clear that we now have $2m$-frequency oscillations ($2m \leqq n$). We obtain

$$\frac{dA_{2r-1}^\circ}{\varepsilon\, dt} = -H_r^\circ A_{2r}^{\circ 2} \sin\lambda_r, \qquad \frac{dA_{2r}^\circ}{\varepsilon\, dt} = 2H_r^\circ A_{2r}^\circ A_{2r-1}^\circ \sin\lambda_r \tag{4.1}$$

$$\frac{d\varphi_{2r-1}}{\varepsilon dt} = \alpha_{2r-1,\,2r-1} A_{2r-1}^{\circ 2} + 2\sum_{s=1}^{2m}{}' \alpha_{s,\,2r-1} A_s^{\circ 2} + \sum_{s=1}^{m} \beta_{s,\,2r-1} H_s^2 - \frac{H_r^\circ A_{2r}^{\circ 2}}{A_{2r-1}^\circ}\cos\lambda_r$$

$$\frac{d\varphi_{2r}}{\varepsilon dt} = \alpha_{2r,\,2r} A_{2r}^{\circ 2} + 2\sum_{s=1}^{2m}{}' \alpha_{s,\,2r} A_s^{\circ 2} + \sum_{s=1}^{m} \beta_{s,\,2r} H_s^2 - 2H_r^\circ A_{2r-1}^\circ \cos\lambda_r$$

$$\lambda_r = (2\omega_{2r} - p_r - \omega_{2r-1})\, t - 2\varphi_{2r} + \psi_r + \varphi_{2r-1}$$

$$H_s^{\circ 2} = A_s^2 m_s \omega_s, \quad H_r^\circ = \frac{1}{16}\, \bar g_{2r-1,\,2r}^{(2r,\,n+r)} (\omega_{2r}, \omega_{2r}, -p_r)\, H_r\, (m_{2r-1}\omega_{2r-1})^{-1/2} (m_{2r}\omega_{2r})^{-1}$$

$$\alpha_{sk} = -\frac{g_{kk}^{(ss)}(\omega_k, \omega_s, -\omega_s)}{16 m_k \omega_k m_s} = \alpha_{ks}, \qquad \beta_{sk} = \frac{\bar g_{kk}^{(n+s,\,n+s)}(\omega_k, p_s, -p_s)}{84 m_k \omega_k}$$

The first two of Eqs. (4.1) have the integral

$$2A_{2r-1}^{\circ 2} + A_{2r}^{\circ 2} = \varkappa^2 \tag{4.2}$$

where $\varkappa^2$ is a constant of integration. Taking into account the expression for λ_r and the integral (4.2), we obtain the canonical system

$$\frac{dA_{2r-1}^\circ}{\varepsilon dt} = -H_r^\circ (\varkappa^2 - 2H_{2r-1}^{\circ 2}) \sin\lambda_r, \quad r = 1, 2, \ldots, m \tag{4.3}$$

$$\frac{d\lambda_r}{\varepsilon dt} = 2m_r^\circ + 4a_{2r-1,\,2r-1}^\circ A_{2r-1}^{\circ 2} + 2\sum_{s=1}^{2m}{}'' a_{sr}^\circ A_s^{\circ 2} - \frac{H_r^\circ}{A_{2r-1}^\circ}(\varkappa^2 - 6A_{2r-1}^{\circ 2})\cos\lambda_r$$

$$2m_r^\circ = \frac{2\omega_{2r} - p_r - \omega_{2r-1}}{\varepsilon} + \sum_{s=1}^{m} (\beta_{s,\,2r} - 2\beta_{s2r-1}) H_s^2 + 2(\alpha_{2r,\,2r-1} - \alpha_{2r,\,2r})\varkappa^2$$

$$4a_{2r-1,\,2r-1}^\circ = \alpha_{2r-1,\,2r-1} - 8\alpha_{2r-1,\,2r} + 4\alpha_{2r,\,2r}$$

$$a_{sr}^\circ = \alpha_{s,\,2r-1} - 2\alpha_{s,\,2r}, \quad s \neq 2r-1,\, 2r, \quad a_{sr}^\circ \neq a_{rs}^\circ$$

The double prime on the summation sign here means that the terms with the subscripts $2r-1$ and $2r$ are omitted.

B. The equalities

$$\omega_{2r-1} \approx 2p_r - \omega_{2r}, \quad \omega_{2r} \approx 2p_r - \omega_{2r-1} \quad r = 1, 2, \ldots, m$$

are satisfied. As we did for the subcase A, we obtain the canonical system

$$\frac{dA_{2r-1}^\circ}{\varepsilon dt} = -H_r^\circ (A_{2r-1}^{\circ 2} - \varkappa)^{1/2} \sin\lambda_r, \quad r = 1, 2, \ldots, m \tag{4.4}$$

$$\frac{d\lambda_r}{\varepsilon dt} = 2m_r^\circ + 4a_{2r-1,\,2r-1}^\circ A_{2r-1}^{\circ 2} + 2\sum_{s=1}^{2m}{}'' a_{sr}^\circ A_s^{\circ 2} - \frac{H_r^\circ(-\varkappa + 2A_{2r-1}^{\circ 2})}{A_{2r-1}^\circ (A_{2r-1}^{\circ 2} - \varkappa)^{1/2}}\cos\lambda_r$$

$$2m_r^\circ = \frac{2p_r - \omega_{2r} - \omega_{2r-1}}{\varepsilon} + \sum_{s=1}^{m} (\beta_{s,\,2r-1} + \beta_{s,\,2r}) H_s^2 - (\alpha_{2r,\,2r} + 2\alpha_{2r,\,2r-1})\varkappa$$

$$H_r{}^\circ = \frac{1}{16}\,\bar{g}^{(n+r,\,n+r)}_{2r-1,\,2r}(p_r,\,p_r-\omega_{2r})\,H_r{}^2\,(m_{2r}\omega_{2r}m_{2r-1}\omega_{2r-1})^{1/2}$$
$$\lambda_r = (2p_r - \omega_{2r-1} - \omega_{2r})\,t - 2\psi_r + \varphi_{2r-1} + \varphi_{2r}$$
$$4\overset{\circ}{a}_{2r-1,\,2r-1} = \alpha_{2r-1,\,2r-1} + 4\alpha_{2r,\,2r-1} + \alpha_{2r,\,2r}$$
$$a_{sr}{}^\circ = \alpha_{s,\,2r-1} + \alpha_{s,\,2r}, \quad s \neq 2r-1,\,2r, \quad a_{sr}{}^\circ \neq a_{rs}{}^\circ$$

Here $\varkappa$ is a constant of integration.

With a slight change in notation, we can bring the subcases defined by the equalities

$$\pm\,\omega_{2r} \approx 2\omega_{2r-1} \pm p_r, \quad \omega_{2r-1} \approx -\,\omega_{2r-1} \pm \omega_{2r} \mp p_r, \quad r = 1, 2, \ldots m$$

either to the subcase A or to the subcase B.

The subcases A and B are essentially different. The integral (4.2) is typical for an internal resonance. It proves to be the case that external perturbations produce a specific type of resonance having certain features in common with an internal resonance.

C. The equalities

$$\omega_{2r-1} \approx \omega_{2r} + p_{2r-1} - p_{2r}, \quad \omega_{2r} \approx \omega_{2r-1} + p_{2r} - p_{2r-1}, \quad r = 1, 2, \ldots, m$$

are satisfied. The canonical system for this subcase has the form

$$d\overset{\circ}{A}_{2r-1}/\varepsilon dt = -\,H_r{}^\circ\,(\varkappa^2 - \overset{\circ 2}{A}_{2r-1})^{1/2}\sin\lambda_r, \quad r = 1, 2, \ldots, m \tag{4.5}$$
$$\frac{d\lambda_r}{\varepsilon dt} = 2m_r{}^\circ + 4\overset{\circ}{a}_{2r-1,\,2r-1}\overset{\circ 2}{A}_{2r-1} + 2\sum_{s=1}^{2m}{}'' a_{sr}{}^\circ A_s{}^{\circ 2} - \frac{H_r{}^\circ\,(\varkappa^2 - 2\overset{\circ 2}{A}_{2r-1})}{\overset{\circ}{A}_{2r-1}\,(\varkappa^2 - \overset{\circ 2}{A}_{2r-1})^{1/2}}\cos\lambda_r$$

$$m_r{}^\circ = \frac{\omega_{2r} + p_{2r-1} - \omega_{2r-1} - p_{2r}}{\varepsilon} + \sum_{s=1}^{m}(\beta_{s,\,2r-1} - \beta_{s,\,2r})\,H_s{}^2 + (2\alpha_{2r,\,2r-1} - \alpha_{2r,\,2r})\,\varkappa^2$$
$$H_r{}^\circ = \frac{1}{24}\,\bar{g}^{(n+2r,\,n+2r-1)}_{2r,\,2r-1}(\omega_{2r},\,p_{2r-1},\,-p_{2r}-p_{2r})\,H_{2r}H_{2r-1}\,(\omega_{2r}m_{2r}\omega_{2r-1}m_{2r-1})^{1/2}$$
$$\lambda_r = (\omega_{2r} + p_{2r-1} - \omega_{2r-1} - p_{2r})\,t - \varphi_{2r} - \psi_{2r-1} + \varphi_{2r-1} + \psi_{2r}$$
$$4\overset{\circ}{a}_{2r-1,\,2r-1} = \alpha_{2r-1,\,2r-1} - 4\alpha_{2r-1,\,2r} + \alpha_{2r,\,2r}, \quad a_{sr}{}^\circ = \alpha_{s,\,2r-1} - \alpha_{s,\,2r}, \quad s \neq 2r-1,\,2r$$

Other subcases are also possible; however, they lead to the canonical systems already considered.

5. In the third fundamental case three characteristic frequencies are involved in the relationship defining resonance. We identify several subcases.

A. The equalities

$$\omega_{3r-2} \approx \omega_{3r-1} + \omega_{3r} - p_r, \quad \omega_{3r-1} \approx \omega_{3r-2} - \omega_{3r} + p_r,$$
$$\omega_{3r} \approx \omega_{3r-2} - \omega_{3r-1} + p_r, \quad r = 1, 2, \ldots, m$$

are satisfied. It is clear that here we have $3m$-frequency oscillations ($3m \leqq n$). We now obtain the system

$$\frac{d\overset{\circ}{A}_{3r-2}}{\varepsilon dt} = -\,H_r{}^\circ\overset{\circ}{A}_{3r-1}A_{3r}{}^\circ\sin\lambda_r \tag{5.1}$$
$$\frac{d\overset{\circ}{A}_{3r-1}}{\varepsilon dt} = 2H_r{}^\circ\overset{\circ}{A}_{3r-2}A_{3r}{}^\circ\sin\lambda_r, \qquad \frac{dA_{3r}{}^\circ}{\varepsilon dt} = 2H_r{}^\circ\overset{\circ}{A}_{3r-2}\overset{\circ}{A}_{3r-1}\sin\lambda_r$$

$$\frac{d\varphi_{3r-2}}{\varepsilon dt} = \alpha_{3r-2,\,3r-2}\overset{\circ 2}{A}_{3r-2} + 2\sum_{s=1}^{3m}{}'\alpha_{s,\,3r-2}A_s{}^{\circ 2} + \sum_{s=1}^{m}\beta_{s,\,3r-2}H_s{}^2 - \frac{H_r{}^\circ\overset{\circ}{A}_{3r-1}A_{3r}{}^\circ}{\overset{\circ}{A}_{3r-2}}\cos\lambda_r$$

$$\frac{d\varphi_{3r-1}}{\varepsilon dt} = \alpha_{3r-1,\,3r-1}\overset{\circ}{A}{}^{2}_{3r-1} + 2\sum_{s=1}^{3m}{}' \alpha_{s,\,3r-1}\overset{\circ}{A}{}_s^{2} + \sum_{s=1}^{m} \beta_{s,\,3r-1}H_s^2 - \frac{H_r^\circ \overset{\circ}{A}_{3r-2}\overset{\circ}{A}_{3r}}{\overset{\circ}{A}_{3r-1}}\cos\lambda_r$$

$$\frac{d\varphi_{3r}}{\varepsilon dt} = \alpha_{3r,\,3r}\overset{\circ}{A}{}_{3r}^{2} + 2\sum_{s=1}^{3m}{}' \alpha_{s,\,3r}\overset{\circ}{A}{}_s^{2} + \sum_{s=1}^{m} \beta_{s,\,3r}H_s^2 - \frac{H_r^\circ \overset{\circ}{A}_{3r-2}\overset{\circ}{A}_{3r-1}}{\overset{\circ}{A}_{3r}}\cos\lambda_r$$

$$H_r^\circ = \frac{1}{16}\bar{g}^{(3r,\,n+r)}_{3r-2,\,3r-1}(\omega_{3r-1}, \omega_{3r}, -p_r)H_r(\omega_{3r-2}m_{3r-2}\omega_{3r-1}m_{3r-1}\omega_{3r}m_{3r})^{-1/2}$$

$$\lambda_r = (\omega_{3r-1} + \omega_{3r} - p_r - \omega_{3r-2})t - \varphi_{3r-1} - \varphi_{3r} + \psi_r + \varphi_{3r-2}$$

The first three of Eqs. (5.1) have the integrals

$$4\overset{\circ}{A}{}^{2}_{3r-2} + \overset{\circ}{A}{}^{2}_{3r-1} + \overset{\circ}{A}{}_{3r}^{2} = \varkappa^2, \quad 2\overset{\circ}{A}{}^{2}_{3r-2} + \overset{\circ}{A}{}^{2}_{3r-1} = \varkappa_1^2 \tag{5.2}$$

Here $\varkappa^2$ and $\varkappa_1^2$ are constants of integration. Taking into account the expressions for λ_r and the integrals (5.2), we obtain the canonical system

$$\frac{d\overset{\circ}{A}_{3r-2}}{\varepsilon dt} = -H_r^\circ(\varkappa_1^2 - 2\overset{\circ}{A}{}^{2}_{3r-2})^{1/2}(\varkappa^2 - \varkappa^{12} - 2\overset{\circ}{A}{}^{2}_{3r-2})^{1/2}\sin\lambda_r, \quad r = 1, 2, \ldots, m \tag{5.3}$$

$$\frac{d\lambda_r}{\varepsilon\, dt} = 2m_r^\circ + 4\overset{\circ}{a}_{3r-2,\,3r-2}\overset{\circ}{A}{}^{2}_{3r-2} + 2\sum_{s=1}^{3m}{}''' a_{sr}^\circ \overset{\circ}{A}{}_s^{2} -$$

$$\frac{H_r^\circ}{\overset{\circ}{A}_{3r-2}}[\varkappa_1^2(\varkappa^2 - \varkappa_1^2) - 4\overset{\circ}{A}{}^{2}_{3r-2}(\varkappa^2 - \varkappa_1^2) + 12\overset{\circ}{A}{}^{4}_{3r-2}] \times$$

$$(\varkappa_1^2 - 2\overset{\circ}{A}{}^{2}_{3r-2})^{-1/2}(\varkappa^2 - \varkappa_1^2 - 2\overset{\circ}{A}{}^{2}_{3r-2})^{-1/2}\cos\lambda_r$$

$$2m_r^\circ = \frac{\omega_{3r-1} + \omega_{3r} - p_r - \omega_{3r-2}}{\varepsilon} + \sum_{s=1}^{m}(\beta_{s,\,3r-2} - \beta_{s,\,3r-1} - \beta_{s,\,3r})H_s^2 +$$

$$(2\alpha_{3r-1,\,3r-2} - \alpha_{3r-1,\,3r-1} - 2\alpha_{3r-1,\,3r})\varkappa_1^2 + (2\alpha_{3r,\,3r-2} - 2\alpha_{3r,\,3r-1} - \alpha_{3r,\,3r})\varkappa^2$$

$$4\overset{\circ}{a}_{3r-2,\,3r-2} = \alpha_{3r-2,\,3r-2} - 6\alpha_{3r-2,\,3r-1} + 2\alpha_{3r-1,\,3r-1} + 6\alpha_{3r-1,\,3r} - 4\alpha_{3r,\,3r-2} + 2\alpha_{3r,\,3r}$$

$$a_{sr}^\circ = \alpha_{s,\,3r-2} - \alpha_{s,\,3r-1} - \alpha_{s,\,3r}, \quad s \neq 3r-2,\ 3r-1,\ 3r,\ a_{sr}^\circ \neq a_{rs}^\circ$$

Here the triple prime on the summation sign means that the terms with the subscripts $3r-2$, $3r-1$ and $3r$ are omitted. Certainly, we can eliminate from the integrals (5.2) instead of the amplitudes $\overset{\circ}{A}_{3r-1}$ and A_{3r}°, the amplitudes $\overset{\circ}{A}_{3r-2}$ and A_{3r}° or the amplitudes $\overset{\circ}{A}_{3r-2}$ and $\overset{\circ}{A}_{3r-1}$. However, no essential difference in the resulting canonical systems is obtained.

B. The equalities

$$\omega_{3r} \approx p_r - \omega_{3r-2} - \omega_{3r-1}, \quad \omega_{3r-1} \approx p_r - \omega_{3r-2} - \omega_{3r}, \quad \omega_{3r-2} \approx p_r - \omega_{3r-1},$$
$$r = 1, 2, \ldots, m$$

are satisfied. Similarly to subcase A, we obtain here the canonical system

$$\frac{d\overset{\circ}{A}_{3r-2}}{\varepsilon dt} = H_r^\circ(\overset{\circ}{A}{}^{2}_{3r-2} - \varkappa_1)^{1/2}(\overset{\circ}{A}{}^{2}_{3r-2} - \varkappa + \varkappa_1)^{1/2}\sin\lambda_r, \quad r = 1, 2, \ldots, m \tag{5.4}$$

$$\frac{d\lambda_r}{\varepsilon dt} = 2m_r^\circ - 4\overset{\circ}{a}_{3r-2,\,3r-2}\overset{\circ}{A}{}^{2}_{3r-2} - 2\sum_{s=1}^{3m}{}''' a_{sr}^\circ \overset{\circ}{A}{}_s^{2} +$$

$$\frac{H_r^\circ}{A^\circ_{3r-2}}\,[\varkappa_1(\varkappa-\varkappa_1)-2\varkappa A^{\circ 4}_{3r-2}+3A^{\circ 2}_{3r-2}]\,(A^{\circ 2}_{3r-2}-\varkappa_1)^{-1/2}\times$$
$$(A^{\circ 2}_{3r-2}-\varkappa+\varkappa_1)^{-1/2}\cos\lambda_r$$

$$2m_r^\circ=\frac{\omega_{3r-2}+\omega_{3r-1}+\omega_{3r}-p_r}{\varepsilon}-\sum_{s=1}^{m}(\beta_{s,3r-1}+\beta_{s,3r-2}+\beta_{s,3r})H_s^2+$$
$$\varkappa_1(2\alpha_{3r-1,3r-2}+\alpha_{3r-1,3r-1}+2\alpha_{3r-1,3r})+$$
$$(\varkappa-\varkappa_1)(2\alpha_{3r,3r-2}+2\alpha_{3r-1,3r}+\alpha_{3r,3r})$$

$$\lambda_r=(\omega_{3r-2}+\omega_{3r-1}+\omega_{3r}-p_r)\,t-\varphi_{3r-2}-\varphi_{3r-1}-\varphi_{3r}+\psi_r$$

$$4a^\circ_{3r-2,3r-2}=\alpha_{3r-2,3r-2}+\alpha_{3r-1,3r-1}+\alpha_{3r,3r}+4\alpha_{3r-2,3r-1}+4\alpha_{3r-1,3r}+4\alpha_{3r,3r-2}$$

$$a_{sr}^\circ=\alpha_{s,3r-2}+\alpha_{s,3r-1}+\alpha_{s,3r},\quad s\neq 3r-2,\ 3r-1,\ 3r$$

where $\varkappa$ and $\varkappa_1$ are constants of integration.

Other subcases are also possible; however, for certain differences in notation, they all lead to the canonical systems (5.3) and (5.4).

The three fundamental cases we have considered are characterized by separate isolated groups of equations. We note, in this regard, that even in the canonical systems the equations are grouped in a certain way. When $m=1$, all the canonical systems may be integrated. The corresponding integrals have the form

$$m_1^0A_1^0+{}^3/_4\alpha_{11}A_1^4-A_1^3\cos\lambda_1=c_{3.2} \qquad (5.5)$$
$$m_1^0A_1^2-{}^1/_4\alpha_{11}A_1^4+A_1\cos\lambda_1=c_{3.3}$$
$$m_1^0A_1^{02}-a_{11}^0A_1^{04}-H_1^0A_1^0(\varkappa^2-2A_1^{02})\cos\lambda_1=c_{4.3}$$
$$m_1^0A_1^{02}+a_{11}^0A_1^{04}-H_1^0A_1^0(A_1^{02}-\varkappa)^{1/2}\cos\lambda_1=c_{4.4}$$
$$m_1^0A_1^{02}+z_{11}^0A_1^{04}-H_1^0A_1^0(\varkappa^2-A_1^{02})^{1/2}\cos\lambda_1=c_{4.5}$$
$$m_1^0A_1^{02}+a_{11}^0A_1^{04}-H_1^0A_1^0(\varkappa_1^2-2A_1^{02})^{1/2}(\varkappa^2-\varkappa_1^2-2A_1^{02})^{1/2}\cos\lambda_1=c_{5.3}$$
$$m_1^0A_1^{02}-a_{11}^0A_1^{04}+H_1^0A_1^0(A_1^{02}-\varkappa_1)^{1/2}(A_1^{02}-\varkappa+\varkappa_1)^{1/2}\cos\lambda_1=c_{5.4}$$

The number of the canonical system is indicated here by the subscript of the constant of integration. Further, we can express $A_1^{\circ 2}$ in terms of elliptic functions of the time εt. The study of all the canonical systems for $m=1$ can be carried out on the phase plane, where A_1° and λ_1 are polar coordinates. The phase trajectories are determined by the expressions (5.5). The phase trajectory plots give a complete representation for all possible motions in the systems.

6. We merely note the following three fundamental cases without going into a detailed discussion. They lead to substantially more complex canonical systems, systems which are not integrable for $m=1$. The study of these systems can be carried out for certain particular cases.

The fourth fundamental case is defined by the equalities

$$\omega_r\approx 3p_r,\quad p_{r+1}\approx 3\omega_r,\quad r=1,2,\ldots,m$$

The canonical system for this case consists of $3m$-equations with m unknown amplitudes A_r and $2m$ variables λ_i. The situation here may be linked to combining the subcases A and B of the first fundamental case. The frequencies ω_i and p_j form a sequence in which any two successive terms are connected by a relationship typical for the first fundamental case.

In the fifth fundamental case the frequencies $\omega_i (i = 1, 2, \ldots, 2m)$ and $p_j (j = 1, 2, \ldots, m)$ or $(j = 1, 2, \ldots, 2m)$ form a sequence in which any three (or four) successive terms are connected by a relationship typical for the second fundamental case. The canonical systems for this case consist of $5m - 2$ (or $6m - 3$) equations with $2m$ unknown amplitudes A_i and $3m - 2$ (or $4m - 3$) variables λ_i ($2m$-frequency oscillations).

In the sixth fundamental case the frequencies $\omega_i (i = 1, 2, \ldots 3m)$ and $p_j (j = 1, 2, \ldots, m)$, form a sequence in which any four successive terms are connected by a relationship typical for the third fundamental case. The canonical systems in this case consist of $7m - 3$ equations with $3m$ unknown amplitudes A_r and $4m - 3$ variables λ_i ($3m$ -frequency oscillations).

The consideration of these fundamental resonance cases does not exhaust all the cases possible. They merely indicate the main directions in which to proceed in studying possible types of resonances.

7. We now consider a case in which the equalities (1.4) are not satisfied. Thus we seek a solution of the system (1.4), not in the form (2.1) but in the form

$$q_k = \sum_{s=1}^{N} M_k^s \cos(p_s t - \psi_s) + \sum_{s=1}^{m} L_k^{(s)} A_s \cos(\omega_s t - \varphi_s) + \varepsilon q_{k1} + \varepsilon^2 q_{k2} + \ldots, \quad k = 1, 2, \ldots, m \tag{7.1}$$

$$M_k^{(s)} = \sum_{j=1}^{n} L_k^{(j)} \frac{\sum_{i=1}^{n} L_i^{(j)} H_s (a_{i,n+s} p_s^2 - c_{i,n+s})}{m_j (\omega_j^2 - p_s^2)}, \quad \omega_j \neq p_s$$

However, we can write the solution (7.1) as follows:

$$q_k = \sum_{s=1}^{m+N} L_k^{(s)} A_s \cos(\omega_s t - \varphi_s) + \varepsilon q_{k1} + \varepsilon^2 q_{k2} + \ldots \tag{7.2}$$

$$L_k^{(s)} = M_k^{(s-m)}, \quad A_s = 1, \quad \omega_s = p_{s-m}, \quad \varphi_s = \psi_{s-m} \quad \text{for } s > m$$

It is now clear that the substitution of (7.2) into the system (1.3), and the subsequent transformations, again lead to the system (2.4) in which the coefficients in the expressions for the F_r will be more complex. This yields no new qualitative results in comparison with the results obtained earlier.

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ON CERTAIN INDICATIONS OF STABILITY WITH TWO LIAPUNOV FUNCTIONS

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We use a Liapunov function with a derivative of constant signs to analyze the problem of asymptotic stability and of instability of an unperturbed motion. We generalize two theorems due to Matrosov [1] for a system of equations of perturbed motion, the right-hand sides of which depend indefinitely on time. The results obtained are also formulated with respect to a part of the variables.

1. Let the following system of equations of perturbed motion be given:

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}) \qquad (\mathbf{X}(t, \mathbf{0}) \equiv 0)$$
$$\mathbf{x} = (x_1, \ldots, x_n) \in R^n, \qquad \|\mathbf{x}\| = (x_1^2 + \ldots + x_n^2)^{1/2} \tag{1.1}$$

where the vector function $\mathbf{X}(t, \mathbf{x})$ is defined and continuous on the set

$$\Gamma = \{(t, \mathbf{x}) : t \geqq 0, \|\mathbf{x}\| < H\} \qquad (0 < H \leqq \infty)$$

while the solutions $\mathbf{x} = \mathbf{x}(t;\ t_0,\ \mathbf{x}_0)$ are defined for $t \geqq t_0$ provided that the initial values $\mathbf{x}_0 = \mathbf{x}(t_0;\ t_0,\ \mathbf{x}_0)$ are sufficiently small in the norm and $t_0 \geqq 0$. Let $\mathbf{x}, \mathbf{y} \in R^n$ and $M \subset R^n$. We introduce the following notation:

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i, \quad \rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|, \quad \rho(\mathbf{x}, M) = \inf\{\rho(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in M\}$$

Definition 1.1. [1]. Let $M \subset R^n$ and the function $U(t, \mathbf{x})$ be defined and continuous on the set

$$\Gamma' = \{(t, \mathbf{x}) : t \geqq 0, \|\mathbf{x}\| \leqq H'\} \qquad (0 < H' = \text{const} < H)$$

We shall consider that $U(t, \mathbf{x})$ is definitely nonzero ($U(t, \mathbf{x}) \neq 0$) in the set $\{(t, \mathbf{x}) : (t, \mathbf{x}) \in \Gamma',\ \mathbf{x} \in M\}$, if for any α_1, α_2 $(0 < \alpha_1 < \alpha_2 < H')$ positive numbers $\beta_1(\alpha_1, \alpha_2) < \alpha_1$ and $\beta_2(\alpha_1, \alpha_2)$ exist such that

$$|U(t, \mathbf{x})| \geqq \beta_2 \quad \text{for} \quad (t, \mathbf{x}) \in \Gamma', \quad \alpha_1 \leqq \|\mathbf{x}\| \leqq \alpha_2$$
$$\rho(\mathbf{x}, M) \leqq \beta_1$$